به نام خداوند جان و خرد

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**1.** Consider the following linear programming problem with one constraint (a knapsack problem):

$$\begin{gathered} max \quad \sum_{i=1}^{n} c_j x_j, \\ s.t. \ \sum_{j=1}^{n} a_j x_j \le b, \\ x_j \ge 0, \ \forall j = 1, \cdots, n, \end{gathered}$$

where $b > 0, \ c_j \ge 0, \forall j$, and $a_j > 0, \forall j$.

Based on the ratios $\frac{c_j}{a_j}, j = 1, \cdots, n$ present a general rule to slove this problem (Hint: find all BFSs and find an optimium by comparing the BFSs.)

**2.**a) Show that one and only one of the following two systems has a solution:

System 1: $Ax = 0, \qquad x \ge 0, \quad cx > 0,$

System 2: $wA \ge c, \quad w = unrestricted.$

b) Show that the system $Ax \le 0$ and $cx > 0$ with $c = (1, 0, 5)$ and

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

has a feasible solution in $\mathbb{R}^3$. Illustrate geometrically.

**3.** a) Write the KKT optimality conditions for the following problem:

$\min cx$ subject to $\{A_1 x = b_1, \quad A_2 x \ge b_2, \quad x \ge 0\}$.

b) Consider the problem: Minimize $cx$ subject to $Ax \ge b, x \ge 0$. Let $x^*$ be an optimal solution. Suppose that $A$ is decomposed into $[A_1, A_2]^T$ and $b$ is decomposed into $[b_1, b_2]^T$, such that $A_1 x^* = b_1$ and $A_2 x^* > b_2$ . Show that $x^*$ is also an optimal solution to the problem: Minimize $cx$ subject to $A_1 x \ge b_1, x \ge 0$, and to the problem: Minimize $cx$ subject to $A_1 x = b_1, x \ge 0$.

**4.** Use the simplex method for bounded variables to solve the following problem:

$$\begin{gathered} min \quad -2x_1 - x_2 - 3x_3, \\ 3x_1 + x_2 + x_3 \le 12 \\ -x_1 + x_2 \le 4 \\ x_2 + 2x_3 \le 8 \\ 0 \le x_1 \le 3 \end{gathered}$$

$$0 \le x_2 \le 5$$
$$0 \le x_3 \le 4.$$

**5.** Consider the following problem:

$$min \ -x_1 + x_2 - 2x_3$$
$$x_1 + x_2 + 2x_3 \le 6,$$
$$-x_1 + 2x_2 + 3x_3 \le 9$$
$$x_1, x_2, x_3 \ge 0.$$

Suppose that the vector $c = (-1, 1, -2)$ is replaced by $(-1, 1, -2) + \lambda(2, -2, 3)$ where $\lambda$ is a nonnegative number. Using parametric analysis, find optimal solutions for all values of $\lambda$.

**6.** Consider the problem given in Exercise 4.

a) Using sensitivity analysis, find a new optimal solution if $c_3$ is changed from $-2$ to 3.

b). Using sensitivity analysis, find a new optimal solution if $a_3$ is changed from $(2, 3)^t$ to $(-4, 1)^t$.

c). Suppose that the variable $x_3$, is deleted from problem. What happens to the optimal solution.

d) find the optimal values of the dual problem, using the optimal simplex table.

**7.** Consider the problem: Minimize $cx$ subject to $Ax = b, x \ge 0$ where $m = n, c = b^t$ and $A = A^t$. Show that if there exists an $x_0$ such that $Ax_0 = b, x_0 \ge 0$, then $x_0$ is an optimal solution.